Math 1522 - Exam 2 Study Guide

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Contents

Summary and Disclaimer

This is a study guide for the first exam for math 1522 at the University of New Mexico (Calculus II). The exam covers chapter 7 of Stewart's Calculus. As such, this study guide is focused on that material. I assume that the student reading this study guide is familiar with the material from a calculus 1 course, including solving integrals with u-substitution. If a you feel that you need to review this material, you can send me an email, or take a look at Paul's Online Math notes:

https://tutorial.math.lamar.edu/

If you are not in my class, I cannot guarantee how much these notes will help you. With that said, if your TA or instructor has shared these with you, then you will most likely get some use out of them.

Methods and Techniques

This exam deals with methods of integration. We assume that you are familiar with u-substitution, and that the only methods that will be generally tested are the new methods. The first of these is integration by parts, which is in some sense the integral version of the chain rule.

Integration by Parts

Since integrals undo derivatives, we want to find out what happens when we undo the chain rule. This gives us integration by parts, which has the following form.

$$\int u \, dv = uv - \int v \, du.$$

To do this in practice, you pick u to be the thing in the integral that you don't know how to integrate, but that you do know how to differentiate, and you let dv be everything else.

Integration by parts questions on quizzes and exams usually come in three varieties, which we will discuss down in the worked examples section.

Our next integration technique is trigonometric integrals. For these, we need to have the following trigonometric identities.

Useful Trigonometric Identities

First, we have the Pythagorean identity. It is called this because it comes from the definition of sin and cos combined with the Pythagorean Theorem.

$$\sin^2(x) + \cos^2(x) = 1$$

Next, we have two formulas which come from rearranging the double angle identities. They are called this because they tell us how $\cos(2x)$ relates to $\sin^2(x)$ and $\cos^2(x)$.

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

Now we can deal with trigonometric integrals. Trigonometric integrals are integrals involving only trigonometric functions. We can evaluate these with the following general strategy.

Trigonometric Integrals

We wish to evaluate integrals of the form

$$\int \sin^n(x)\cos^m(x)\ dx.$$

If n is odd, we can write this as

$$\int (1-\cos^2(x))^{\frac{n-1}{2}}\cos^m(x)\sin(x)\ dx$$

and substitute in $u = \cos(x)$ to solve. If m is odd, we can write the integral as

$$\int \sin^n(x)(1-\sin^2(x))^{\frac{m-1}{2}}\cos(x)\ dx$$

and set $u = \sin(x)$ to solve.
If m and n are both even, we use the double angle identities to change them both into terms of $\cos(2x)$, and solve that integral.

Sometimes you get trigonometric integrals which aren't entirely integer powers. For instance, you might have $\sqrt{\sin(\theta)}$ somewhere in the integral. For these, you want to choose u to be the function which will be hardest to turn into another function. Here, that would be $\sin(\theta)$. We will work through one of these later.

Next, we want to be able to use our ability to solve trigonometric integrals in order to solve other integrals that we couldn't do previously. In this case, we want to do a clever kind of substitution, called a trigonometric substitution.

Trigonometric Substitution

We would like to solve integrals containing an $x^2 + a^2$ term, $x^2 - a^2$ term, or $a^2 - x^2$ term, where a is some constant number (think, 1, 2, or even $\sqrt{3}$). We do the following substitutions, and solve the resulting integral:

Equation	Substitution
$x^2 + a^2$	$x = a\tan(\theta)$
$x^2 - a^2$	$x = a\sec(\theta)$
$a^2 - x^2$	$x = a\sin(\theta)$

Each of these substitutions has an associated triangle which allows us to switch out of terms of θ at the end of our integration.

For our final integration technique, we need to know how to evaluate improper integrals.

Improper Integrals

We want to evaluate integrals of the form

$$\int_a^b f(x)\ dx$$

where

$$\int f(x)\ dx$$

is undefined at either a or b. Without losing anything, we can always assume that this happens at b. So, we say that

$$\int_a^b f(x)\ dx$$

is defined if

$$\lim t \to b \int_a^t f(x)\ dx$$

is defined.

And lastly, depending on your professor, you may or may not need to know the trapezoid rule for estimating integrals. As the explanation in the book is quite good for this subject, I encourage you to read it there. Otherwise, the following video from the Organic Chemistry Tutor does a good job at explaining it:

https://youtu.be/Rn9Gr52zhrY

Worked Examples

We will now work through some examples, beginning with integration by parts. As previously mentioned, on quizzes and tests, these questions come in three different varieties. The first variety requires you to do integration by parts more than once. The second variety requires you to do integration by parts multiple times in order to get the same integral back again, and then to use algebra to solve for the integral. The final is integration by parts where you let $dv = dx$, and let u be everything else.

We will work through one example of each of these, beginning with the first variety.

Example: Evaluate

$$\int x^3 e^x \, dx$$

We evaluate this integral by letting $u = x^3$ and $dv = e^x \, dx$. Then $v = e^x$ and $du = 3x^2 \, dx$. So,

$$\int x^3 e^x \, dx = x^3 e^x - 3\int x^2 e^x \, dx.$$

We take $dv = e^x \, dx$ and $u = x^2$. This time $v = e^x$ and $du = 2x \, dx$, so we get

$$\int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6\int x e^x \, dx.$$

Finally, we do one last iteration of integration by parts. This time we let $dv = e^x \, dx$ and $u = x$. Then $v = e^x$ and $du = dx$. So, we have

$$\int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6xe^x - 6\int e^x \, dx.$$

And we know this last integral is just $e^x + C$. So,

$$\int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C.$$

Now we will do integration by parts of the second variety.

Example: Evaluate

$$\int \cos(x) e^x \, dx.$$

We begin by setting $dv = e^x \, dx$ and $u = \cos(x)$. Then $v = e^x$ and $du = -\sin(x) \, dx$. So,

$$\int \cos(x) e^x \, dx = \cos(x) e^x + \int \sin(x) e^x \, dx.$$

This time we let $dv = e^x \, dx$ and $u = \sin(x)$. Then $v = e^x$ and $du = \cos(x) \, dx$. So,

$$\int \cos(x)e^x \, dx = \cos(x)e^x + \sin(x)e^x - \int \cos(x)e^x \, dx.$$

Adding the integral to both sides gives us

$$2\int \cos(x)e^x \, dx = \cos(x)e^x + \sin(x)e^x,$$

and dividing by two and adding on a $+C$ gives us our final answer of

$$\int \cos(x)e^x \, dx = \frac{\cos(x)e^x + \sin(x)e^x}{2} + C.$$

Finally, we work through an example of the third variety of integration by parts. You should do this variety when you don't know how to integrate any of the functions in an integral, but when you do know how to differentiate them.

Example: Evaluate

$$\int \ln(x) \, dx.$$

We set $dv = dx$ and $u = \ln(x)$. Then $v = x$ and $du = \frac{dx}{x}$. So,

$$\int \ln(x) \, dx = x\ln(x) - \int \frac{x}{x} \, dx = x\ln(x) - \int 1 \, dx.$$

So,

$$\int \ln(x) \, dx = x\ln(x) - x + C.$$

Practice Problems

1. Evaluate
$$\int x^5 \ln(x)\ dx$$

2. Evaluate the following integral using trigonometric substitution
$$\int \frac{x}{\sqrt{x^2+1}}\ dx$$

3. Evaluate
$$\int_0^{\frac{\pi}{4}} \sin^3(\theta)\sqrt{\cos(\theta)}\ d\theta$$

4. Determine if the integral converges. Evaluate it if it does.
$$\int_1^{\infty} \frac{1}{x^4}\ dx$$

Practice Problem Solutions

1.

Solution: We solve the integral

$$\int x^5 \ln(x)\ dx$$

using integration by parts. We let $u = \ln(x)$ and $dv = x^5\ dx$. Then $du = \frac{dx}{x}$ and $v = \frac{x^6}{6}$. So,

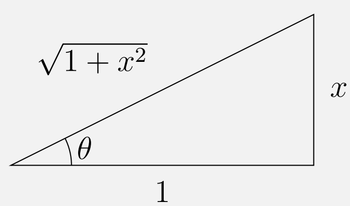
$$\int x^5 \ln(x) = \frac{x^6}{6}\ln(x) - \int \frac{x^6}{6x}\ dx,$$

which we can rewrite as

$$\int x^5 \ln(x) = \frac{x^6}{6}\ln(x) - \int \frac{x^5}{6}\ dx = \frac{x^6}{6}\ln(x) - \frac{x^6}{36} + C.$$

2.

Solution: One may evaluate this integral by a simple u-substitution. But, since we are asked to use a trigonometric substitution, we must do it that way.
Since we see a $1 + x^2$, we will set $x = \tan(\theta)$. This makes $dx = \sec^2(\theta)\ d\theta$. At this point, we will draw out our triangle:



Then, we have that

$$\int \frac{x}{\sqrt{1+x^2}}\ dx = \int \frac{\tan(\theta)}{\sqrt{1+\tan^2(\theta)}} \sec^2(\theta)\ d\theta.$$

And since $1 + \tan^2(\theta) = \sec^2(\theta)$, we have that this is just

$$\int \frac{\tan(\theta)}{\sqrt{\sec^2(\theta)}} \sec^2(\theta) = \int \tan(\theta)\sec(\theta)\ d\theta.$$

And since $\sec'(\theta) = \sec(\theta)\tan(\theta)$, we have that

$$\int \tan(\theta)\sec(\theta)\ d\theta = \sec(\theta) + C.$$

Then, going back to our triangle, we see that in terms of x we have

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \sqrt{1+x^2} + C.$$

Indeed, we may verify that this is the case through the u-substitution. This step is optional, but we will include it here for completeness. We note that if $u = 1 + x^2$, then $du = 2x\,dx$. This would give us

$$\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}2\sqrt{u} + C.$$

Plugging back in our definition of u gives us that

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \sqrt{1+x^2} + C.$$

just like in our trigonometric substitution.

3.

Solution: The first thing that we need to do in order to evaluate this integral is to note that $\cos(\theta)$ will be hard to get out of the square root. So, we want to make $u = \cos(\theta)$. This means that $du = -\sin(\theta)$. So, we want to pull aside one copy of $\sin(\theta)$ and change the rest to $\cos(\theta)$ using $\cos^2(\theta) + \sin^2(\theta) = 1$.
Putting this plan into action gives us

$$\int_0^{\frac{\pi}{4}} \sin^3(\theta)\sqrt{\cos(\theta)}\,d\theta = \int_0^{\frac{\pi}{4}} (1-\cos^2(\theta))\sqrt{\cos(\theta)}\sin(\theta)\,d\theta = -\int_{\theta=0}^{\theta=\frac{\pi}{4}} (1-u^2)\sqrt{u}\,du$$

We now want to change our bounds. If $\theta = 0$, we have that $u = 1$. If $\theta = \frac{\pi}{4}$, then $u = \frac{\sqrt{2}}{2}$. So,

$$-\int_{\theta=0}^{\theta=\frac{\pi}{4}} (1-u^2)\sqrt{u}\,du = -\int_1^{\frac{\sqrt{2}}{2}} (1-u^2)\sqrt{u}\,du = \int_{\frac{\sqrt{2}}{2}}^1 (1-u^2)\sqrt{u}\,du.$$

Finally, we distribute the $\sqrt{u}$ and solve the integral by noting that $\sqrt{u} = u^{\frac{1}{2}}$. This gives us

$$\int_{\frac{\sqrt{2}}{2}}^1 (1-u^2)\sqrt{u}\,du = \int_{\frac{\sqrt{2}}{2}}^1 u^{\frac{1}{2}} - u^{\frac{5}{2}}\,du = \left(\frac{2}{3}\sqrt{u^3} - \frac{7}{2}\sqrt{u^7}\right)_{\frac{\sqrt{2}}{2}}^1$$

And plugging in values gives us that

$$\int_0^{\frac{\pi}{4}} \sin^3(\theta)\sqrt{\cos(\theta)}\ d\theta = \frac{16 - 11\sqrt[4]{2}}{42}$$

4.

Solution: We claim that the integral converges, and we will find its value. The first step to evaluate the integral is to evaluate

$$\int \frac{1}{x^4}\ dx.$$

We note that this is equal to $-\frac{1}{3x^3} + C$. So,

$$\int_1^t \frac{1}{x^4}\ dx = \frac{1}{3} - \frac{1}{3t^3}.$$

So,

$$\lim_{t\to\infty} \int_1^t \frac{1}{x^4}\ dx = \frac{1}{3} - \frac{1}{3t^3} = \frac{1}{3} - 0 = \frac{1}{3}.$$

So,

$$\int \frac{1}{x^4}\ dx = \frac{1}{3}$$

by definition.